Solving Discretely Constrained Mixed Complementarity Problems Using a Median Function with Applications in Energy

Steven A. Gabriel
Professor, University of Maryland, USA
Adjunct Professor, Energy Transition Programme
https://www.ntnu.edu/energytransition, NTNU, Norway

www.stevenagabriel.umd.edu
sgabriel@umd.edu

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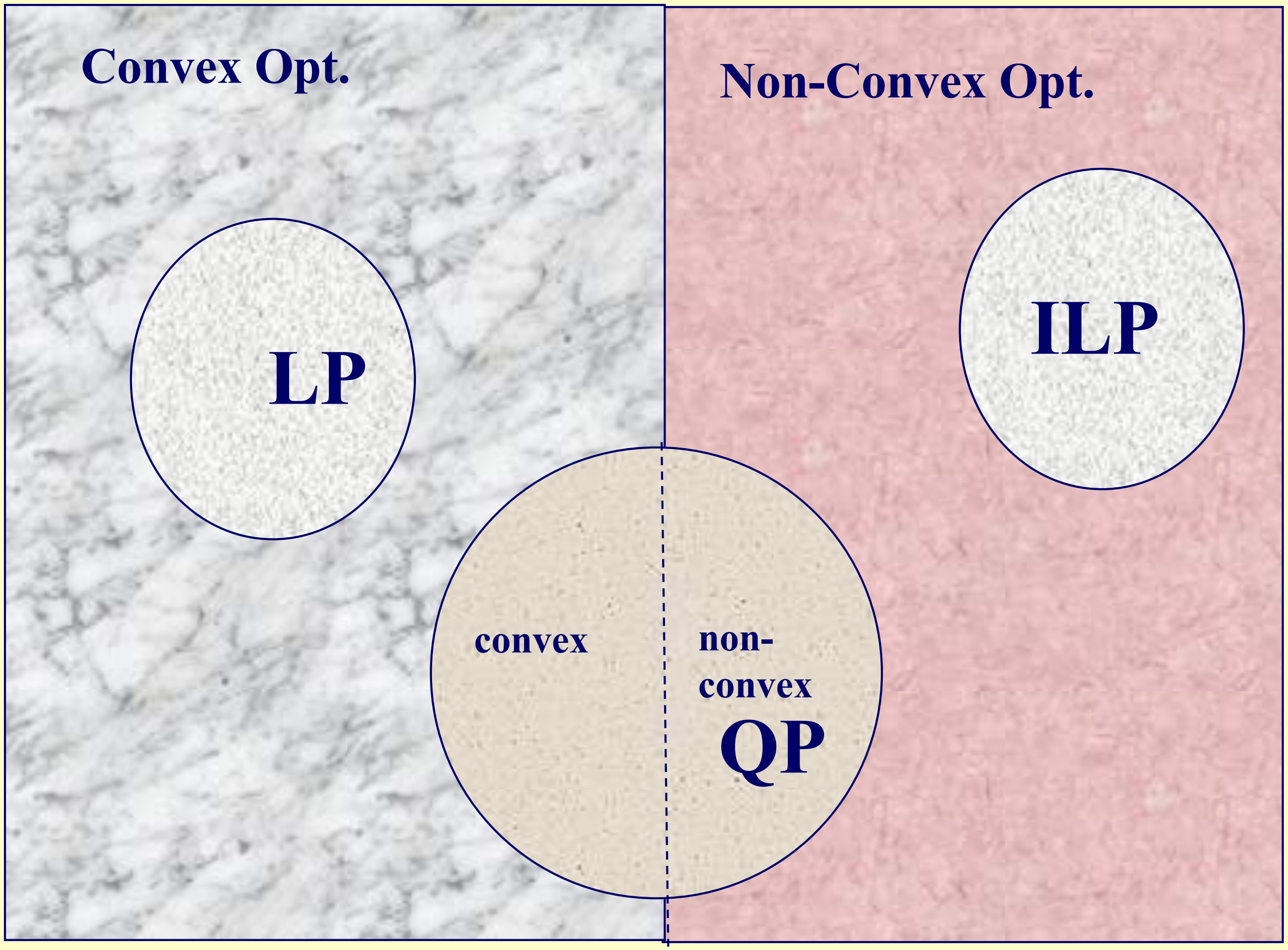
Outline

- Brief introduction to the mixed complementarity problem (MCP) and the bounded MCP
- Problem statement: discretely-constrained MCP (DC-MCP)
- Theoretical results for DC-MCP
- Some Numerical examples:
  - Duopoly in energy production
  - Equity (logic) constraints in network equilibrium problems

Selected DC-MCP References

1. S.A. Gabriel, S. Siddiqui, A.J. Conejo, C. Ruiz, 2013, “Discretely-Constrained, Nash-Cournot Games with an Application to Power Markets,” *Networks and Spatial Economics*, 13(3), 307-326.
2. S.A. Gabriel, A.J. Conejo, C. Ruiz, S. Siddiqui , 2013. "Solving Discretely-Constrained, Mixed Linear Complementarity Problems with Applications in Energy, " *Computers and Operations Research*, 40(5), 1339-1350.
3. S.A. Gabriel, S. Siddiqui, A.J. Conejo, C. Ruiz, 2013, “Discretely-Constrained, Nash-Cournot Games in Energy,” Networks and Spatial Economics, 13(3), 307-326.
4. **S.A. Gabriel, 2017. "Solving Discretely Constrained Mixed Complementarity Problems Using a Median Function," *Optimization and Engineering*, 18(3), 631-658, also preprint at Cahier du GERAD G-2015-123, November 2015.**
5. F. D. Fomeni, S.A. Gabriel, M. J. Anjos, "An RLT Approach for Solving the Binary-Constrained Mixed Linear Complementarity Problem, " in review. Cahiers du GERAD G-2015-60, June 2015, http://wwwold.gerad.ca/en/publications/cahiers.php
6. F. D. Fomeni, S.A. Gabriel, M. J. Anjos, "Applications of Logic Constrained Equilibria to Traffic Networks and to Power Systems with Storage, September 2016 , accepted, *Journal of the Operational Research Society,* February 2018.
7. R. Weinhold and S.A. Gabriel, 2018. “Discretely Constrained Mixed Complementary Problems: Application and Analysis of a Stylized Electricity Market,” accepted at *Journal of the Operational Research Society*, December 2018.

The Big Picture



LP=linear program

ILP=integer linear program

QP=quadratic program

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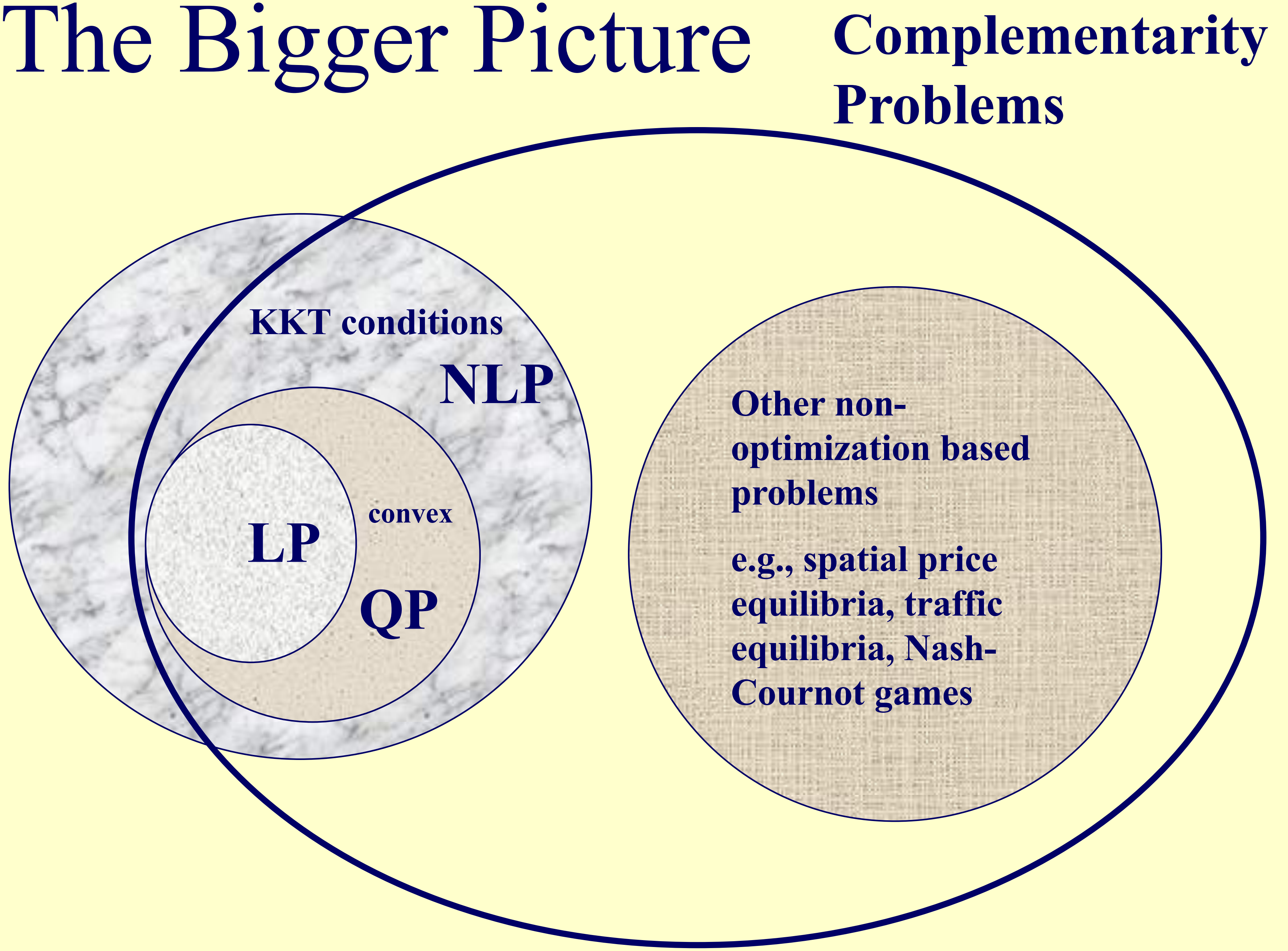
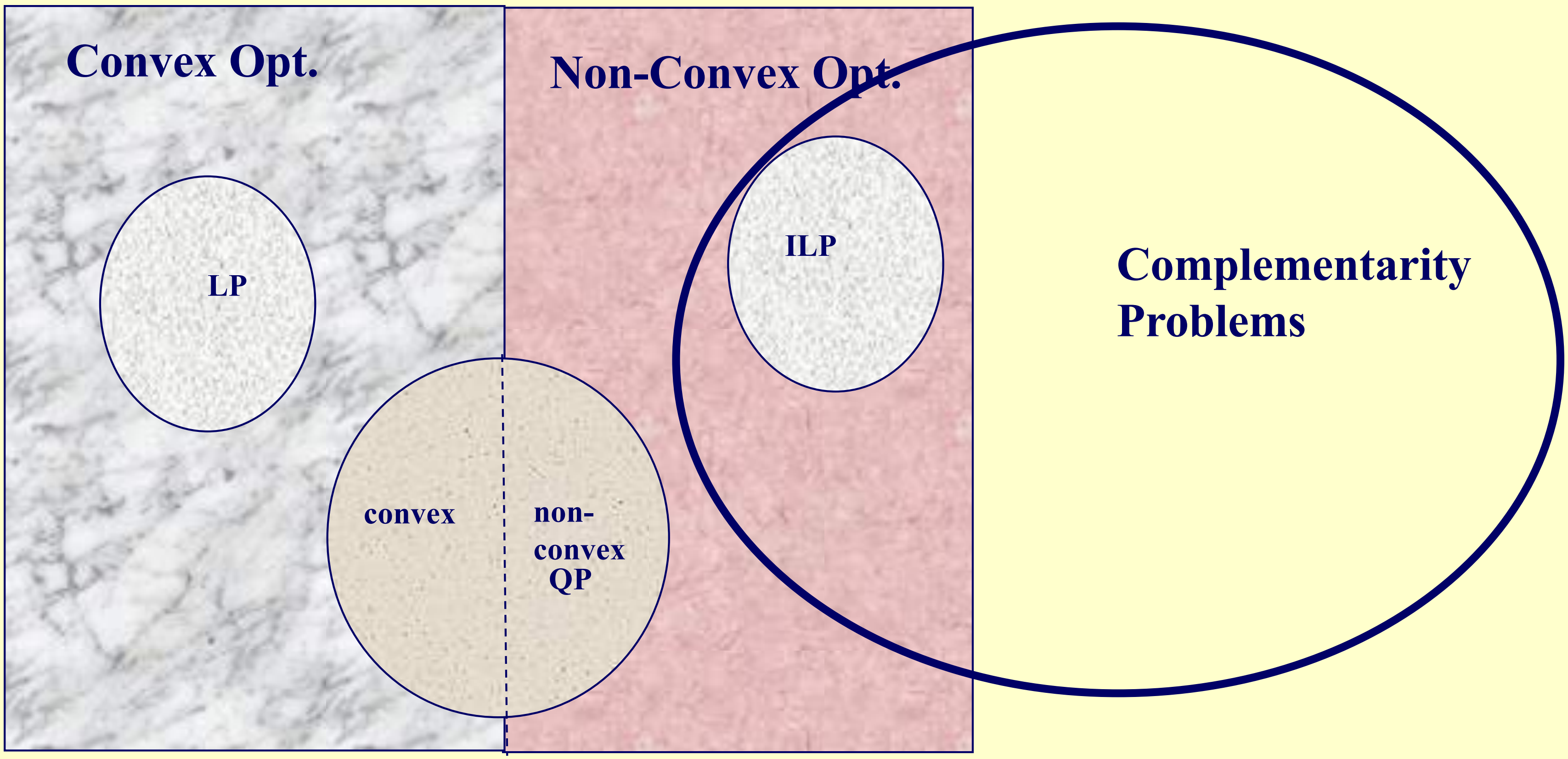
The Bigger Picture



DC-MCP Perspective



LP=linear program

ILP=integer linear program

QP=quadratic program

Equilibrium Problems Expressed as Mixed Nonlinear Complementarity Problems

$\Big($Mixed$\Big)$ Nonlinear Complementarity Problem MNCP (also written as NCP or MCP)

Having a function $F: R^n \to R^n$, find an $x \in R^{n_1}$, $y \in R^{n_2}$ such that

$F_i\left(x,y\right) \geq 0, x_i \geq 0, F_i\left(x,y\right) * x_i = 0 \text{ for } i = 1,\ldots,n_1$

$F_i\left(x,y\right) = 0, y_i$ free, $\text{for } i = n_1+1,\ldots,n$

Example [since all functions (linear) affine --> linear complementarity problem (LCP)]

$$F(x_1,x_2,y_1) = \begin{pmatrix} F_1(x_1,x_2,y_1) \\ F_2(x_1,x_2,y_1) \\ F_3(x_1,x_2,y_1) \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - y_1 \\ x_1 + x_2 + y_1 - 2 \end{pmatrix}$$ so we want to find x_1, x_2, y_1 s.t.

$$x_1 + x_2 \geq 0 \qquad x_1 \geq 0 \qquad \left(x_1 + x_2\right) * x_1 = 0$$

$$x_1 - y_1 \geq 0 \qquad x_2 \geq 0 \qquad \left(x_1 - y_1\right) * x_2 = 0$$

$$x_1 + x_2 + y_1 - 2 = 0 \qquad y_1 \text{ free}$$

One solution: $(x_1, x_2, y_1) = (0,2,0)$, why? Any others?

Energy Producer Nash Game Duopoly Expressed as a Complementarity Problem

-Two producers competing with each other on how much to produce given as $q_i, i=1,2$

- Market Inverse demand function

$$p(q_1+q_2) = \alpha - \beta(q_1+q_2), \text{ where } \alpha,\beta>0$$

that the producers can manipulate by their production

- Production cost function

$$c_i\left(q_i\right) = \gamma_i q_i, i=1,2, \text{ where } \gamma_i>0$$

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Energy Producer Duopoly Expressed as a Complementarity Problem

Producer 1's optimization problem:

$$\max \left(\alpha - \beta(q_1 + q_2)\right) * q_1 - \gamma_1 q_1$$

$$s.t.\ q_1 \geq 0$$

KKT conditions:

Find q_1 s.t. $2\beta q_1 + \beta q_2 - \alpha + \gamma_1 \geq 0 \quad q_1 \geq 0 \quad \left(2\beta q_1 + \beta q_2 - \alpha + \gamma_1\right)\ q_1 = 0$

For Producer 2, similar idea, that is:

Find q_2 s.t. $\beta q_1 + 2\beta q_2 - \alpha + \gamma_2 \geq 0 \quad q_2 \geq 0 \quad \left(\beta q_1 + 2\beta q_2 - \alpha + \gamma_2\right)\ q_2 = 0$

Need to solve both at same time (why?) to get the resulting pure LCP

$$F\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 2\beta q_1 + \beta q_2 - \alpha + \gamma_1 \\ \beta q_1 + 2\beta q_2 - \alpha + \gamma_2 \end{pmatrix}$$

Can generalize to N players, will get a Nash-Cournot equilibrium

Re-expressing the bounded MCP as the zero of a particular median-related function H (Gabriel, 2017)

Given the function $F : R^n \rightarrow R^n$ and vectors $l, u \in R^n \cup \{-\infty, +\infty\}$ with $l \leq u$, consider the mixed complementarity problem (MCP) [8] as follows. Find $x \in R^n$ so that

$$\begin{array}{ll} F_i(x) \geq 0 & x_i = l_i \\ F_i(x) = 0 & l_i < x_i < u_i \\ F_i(x) \leq 0 & x_i = u_i \end{array} \quad (1)$$

Can separate "x" into nonnegative variables x and free variables y to get the conventional MCP

Re-expressing the bounded MCP as the zero of a particular median-related function H (Gabriel, 2017)

$$0 \leq F_i(x,y) \perp x_i \geq 0, i \in I_x = \{1,\ldots,n_x\}$$
$$0 = F_j(x,y), y_j \text{ free}, j \in I_y = \{1,\ldots,n_y\}$$

With additional discrete (integer) restrictions on some of the x or y variables

$$x_d \in Z_+, d \in D_x \subseteq I_x$$
$$y_d \in Z, d \in D_y \subseteq I_y$$

Re-expressing the bounded MCP as the zero of a particular median-related function H (Gabriel, 2016)

Definition 1

A vector pair (x,y) that solves the MCP conditions with the discrete restrictions is a DC-MCP (discretely constrained MCP) solution.

Re-expressing the bounded MCP as the zero of a particular median-related function H

$$H_i(x) = x_i - mid(l_i, u_i, x_i - F_i(x)), \forall i$$

Median-related function H

Note that the function $\|H(x,y)\|$ is in general non-smooth so that (6) is a nonsmooth optimization problem. For example, let $F: R^2 \to R^2$ be defined as $F_1(x,y) = x + y, F_2(x,y) = y$ where $l_1 = 0, u_1 = +\infty,\ l_2 = -\ \infty,\ u_2 = +\infty.$ Then,

$$H(x,y) = \begin{pmatrix} H_1(x,y) \\ H_2(x,y) \end{pmatrix} = \begin{pmatrix} x - mid(0, +\infty, x - (x+y)) \\ y - mid(-\infty, +\infty, y - (y)) \end{pmatrix} = \begin{pmatrix} \begin{cases} x+y & y \le 0 \\ x & y > 0 \end{cases} \\ y \end{pmatrix}$$

So that

$$\|H(x,y)\|_1 = \begin{cases} |x+y| + |y| & y \le 0 \\ |x| + |y| & y > 0 \end{cases}$$
$$= \begin{cases} |x+y| - y & y \le 0 \\ |x| + y & y > 0 \end{cases}$$

thus for $x = 0$ fixed,

$$\|H(0,y)\|_1 = \begin{cases} |y| - y = -2y & y \le 0 \\ y & y > 0 \end{cases}$$

Function H is Nonsmooth

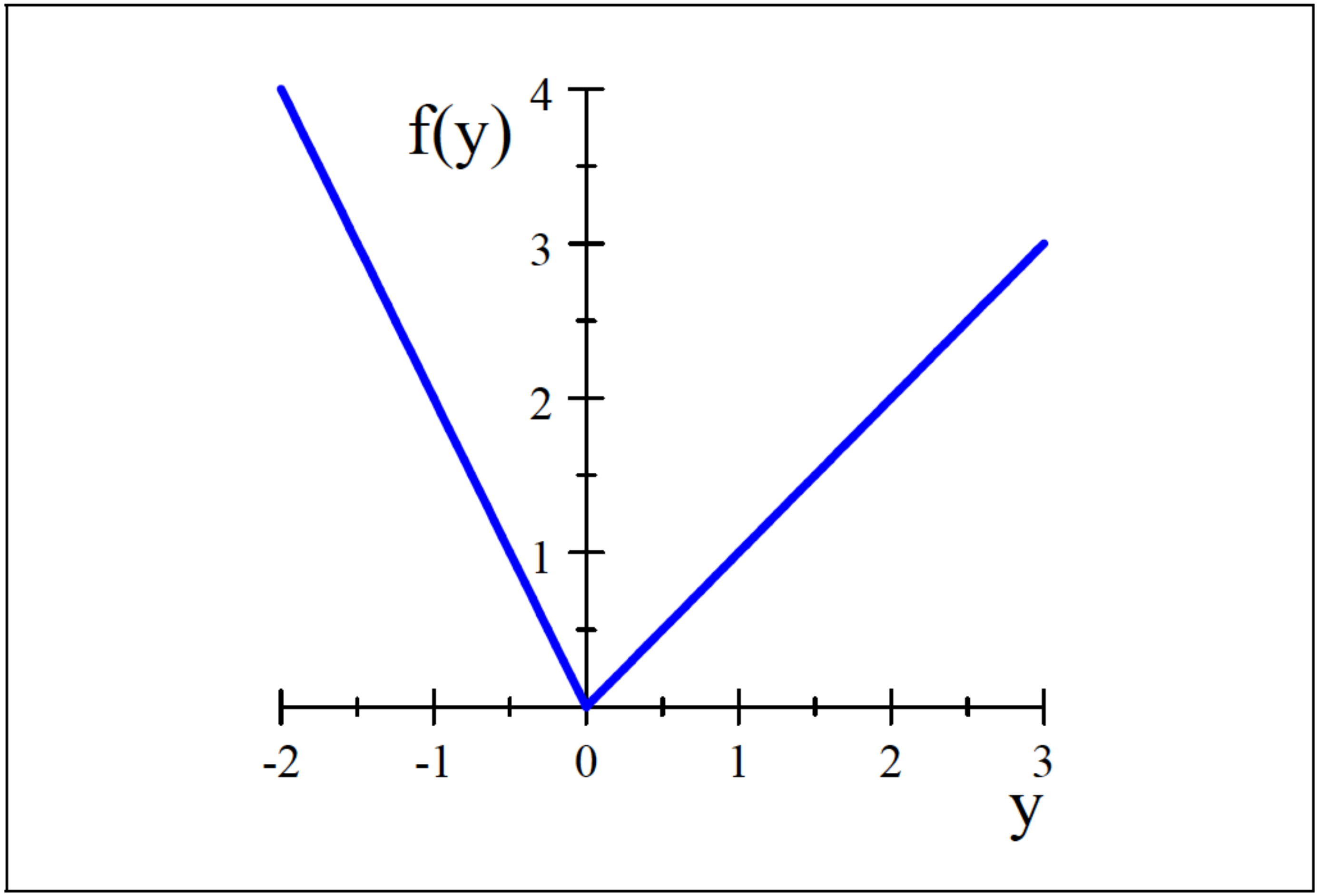


Figure 1: Example of the mid function being non-smooth.

Re-expressing the bounded MCP as the zero of a particular median-related function H

$$H_i(x) = x_i - mid(l_i, u_i, x_i - F_i(x)), \forall i$$

Case 1: $x_i - F_i(x,y) < l_i \leq u_i \Rightarrow z_i = H_i(x,y) = x_i - l_i$

Case 2: $l_i < x_i - F_i(x,y) < u_i \Rightarrow z_i = H_i(x,y) = x_i - (x_i - F_i(x,y)) = F_i(x,y)$

Case 3: $l_i = x_i - F_i(x,y) \leq u_i \Rightarrow z_i = H_i(x,y) = x_i - (x_i - F_i(x,y))$

$= F_i(x,y) = x_i - l_i$

Case 4: $l_i \leq u_i < x_i - F(x,y) \Rightarrow z_i = H_i(x,y) = x_i - u_i$

Case 5: $l_i \leq u_i = x_i - F_i(x,y) \Rightarrow z_i = H_i(x,y) = x_i - u_i = F_i(x,y)$

Main MINLP to solve DC-MCP

Definition 2

A vector pair (x,y) that solves the MINLP below is a relaxed DC-MCP solution.

$$
\begin{aligned}
&\min_{x} \|H(x,y)\| \\
&s.t.\ x_i \in R_+, i \in I_x \backslash D_x \\
&x_i \in Z_+, i \in D_x \\
&y_j \in R\ , j \in I_y \backslash D_y \\
&y_j \in Z, j \in D_y
\end{aligned}
$$

Main MINLP to solve DC-MCP Using L_1 Norm

$$\min_{x,y,z^+,z^-,w^+,w^-,b,\widetilde{b}} f = \sum_{i \in I_x} \left(z_i^+ + z_i^-\right) + \sum_{j \in I_y}^{n} \left(w_j^+ + w_j^-\right)$$

$$s.t. -Mb_i \le x_i - F_i(x,y) - l_i \le M(1-b_i), \forall i \in I_x$$

$$-M\widetilde{b}_i \le x_i - F_i(x,y) - u_i \le M(1-\widetilde{b}_i), \forall i \in I_x$$

$$-M\left(2 - b_i - \widetilde{b}_i\right) \le z_i^+ - z_i^- - x_i + l_i \le M\left(2 - b_i - \widetilde{b}_i\right)$$

$$-M\left(1 + b_i - \widetilde{b}_i\right) \le z_i^+ - z_i^- - F_i(x,y) \le M\left(1 + b_i - \widetilde{b}_i\right)$$

$$-M\left(b_i + \widetilde{b}_i\right) \le z_i^+ - z_i^- - x_i + u_i \le M\left(b_i + \widetilde{b}_i\right)$$

$$w_j^+ - w_j^- = F_j(x,y), \forall j \in I_y$$

$$x_i \in R_+, i \in I_x \backslash D_x$$

$$x_i \in Z_+, i \in D_x$$

$$y_j \in R\ , j \in I_y \backslash D_y$$

$$y_j \in Z, j \in D_y$$

$$b_i, \widetilde{b}_i \in \{0,1\}\ , \forall i \in I_x$$

$$z_i^+, z_i^- \ge 0, \forall i \in I_x$$

$$w_j^+, w_j^- \ge 0, \forall j \in I_y$$

Enforcement of various cases of the mid function, nonnegative variables x

Enforcement of various cases of the mid function, free variables y

Main MINLP to solve DC-MCP Using L_1 Norm

Case 1: $x_i - F_i(x,y) < l_i \le u_i \Rightarrow z_i = H_i(x,y) = x_i - l_i$

Case 2: $l_i < x_i - F_i(x,y) < u_i \Rightarrow z_i = H_i(x,y) = x_i - (x_i - F_i(x,y)) = F_i(x,y)$

Case 3: $l_i = x_i - F_i(x,y) \le u_i \Rightarrow z_i = H_i(x,y) = x_i - (x_i - F_i(x,y))$

$= F_i(x,y) = x_i - l_i$

Case 4: $l_i \le u_i < x_i - F(x,y) \Rightarrow z_i = H_i(x,y) = x_i - u_i$

Case 5: $l_i \le u_i = x_i - F_i(x,y) \Rightarrow z_i = H_i(x,y) = x_i - u_i = F_i(x,y)$

Main MINLP to solve DC-MCP Using L1 Norm

Theorem 1 *For each* $i \in I_x$*, assume that* $l_i < u_i$*. Consider any feasible solution* $\left(x, y, z^+, z^-, w^+, w^-, b, \widetilde{b}\right)$ *to (7). Then, for* $z_i \triangleq z_i^+ - z_i^-, w_j \triangleq w_j^+ - w_j^-$,

$$z_i = H_i\left(x, y\right), \forall i \in I_x$$
$$w_j = H_j\left(x, y\right), \forall j \in I_y$$

Theorem 2 *For each* $i \in I_x$*, assume that* $l_i < u_i$*. Consider any optimal solution* $\left(x^*, y^*, z^{+^*}, z^{-^*}, w^{+^*}, w^{-^*}, b^*, \widetilde{b^*}\right)$ *to (7). Then at most one of* $\left(z_i^{+*}, z_i^{-*}\right)$ *is nonzero and at most one of* $\left(w_i^+, w_i^-\right)$ *is nonzero.*

Theorem 3 *Consider any optimal solution* $\left(x^*, y^*, z^{+^*}, z^{-^*}, w^{+^*}, w^{-^*}, b^*, \widetilde{b^*}\right)$ *to (7) with corresponding optimal objective function value* f^**. Then,*

$f^* = 0 \Leftrightarrow (x^*, y^*)$ *solve the DC-MCP (1), (5).*

Numerical Results

Problem #	n_x	n_y	There is a continuous solution that is integer?	Type of Problem
1a	10	10	yes, by construction	Small Illustrative
1b	10	10	no	Small Illustrative
1c	10	10	no	Small Illustrative
1d	1000	1000	yes, by construction	Large random
1e	1000	1000	yes, but not known in advance	Large random
2a	4	0	yes	Energy duopoly
2b	4	0	no	Energy duopoly
2c1	4	0	no	Energy duopoly
2c2	4	0	no	Energy duopoly
3a	12	0	yes, but not known in advance	Spatial Price Equilibrium
3b	12	0	yes, but not known in advance	Spatial Price Equilibrium
3c	12	0	yes, but not known in advance	Spatial Price Equilibrium

Table 1: Summary of numerical results.

Numerical Example #1: Energy Production, Capacitated Duopoly with Selected Complementarity Relaxation Weighting

$$\max_{q_p} p\left(\sum_p q_p\right) q_p - c_p\left(q_p\right)$$

$$s.t.\ 0 \le q_p \le q_p^{\max} \quad (\lambda_p)$$

Energy Production, Capacitated Duopoly with Selected Complementarity Relaxation Weighting

$$0 \leq \begin{pmatrix} \gamma_1 - \alpha \\ \gamma_2 - \alpha \\ q_1^{\max} \\ q_2^{\max} \end{pmatrix} + \begin{pmatrix} 2\beta & \beta & 1 & 0 \\ \beta & 2\beta & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \perp \begin{pmatrix} q_1 \\ q_2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \geq 0$$

$$\|H(q_1, q_2, \lambda_1, \lambda_2)\|_1 \triangleq \sum_{i=1,4} |H_i(q_1, q_2, \lambda_1, \lambda_2)| = 0 \Leftrightarrow |H_i(q_1, q_2, \lambda_1, \lambda_2)| = 0 \text{ for all } i$$

Energy Production, Capacitated Duopoly with Selected Complementarity Relaxation Weighting

$$0 \leq \begin{pmatrix} \gamma_1 - \alpha \\ \gamma_2 - \alpha \\ q_1^{\max} \\ q_2^{\max} \end{pmatrix} + \begin{pmatrix} 2\beta & \beta & 1 & 0 \\ \beta & 2\beta & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \perp \begin{pmatrix} q_1 \\ q_2 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \geq 0$$

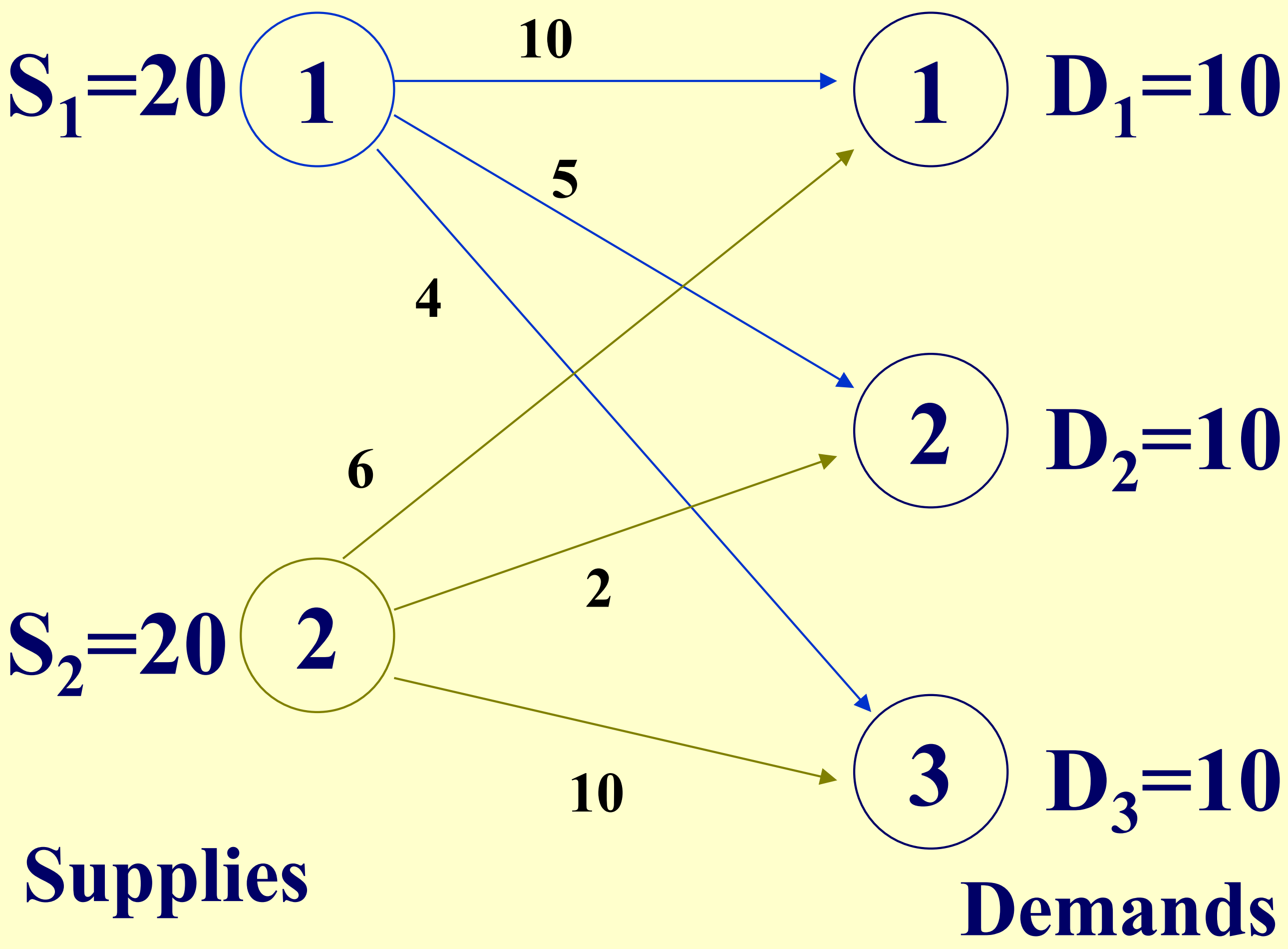
$$\|H(q_1, q_2, \lambda_1, \lambda_2)\|_1 \triangleq \sum_{i=1,4} |H_i(q_1, q_2, \lambda_1, \lambda_2)| = 0 \Leftrightarrow |H_i(q_1, q_2, \lambda_1, \lambda_2)| = 0 \text{ for all } i$$

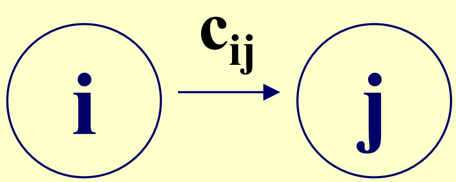
Thus, an equivalent objective function that could be used would be

$$\sum_{i=1,4} \omega_i \, |H_i(q_1, q_2, \lambda_1, \lambda_2)|$$

Numerical Example #2: Spatial Price Equilibrium (SPE) with Equity-Enforcing Constraints

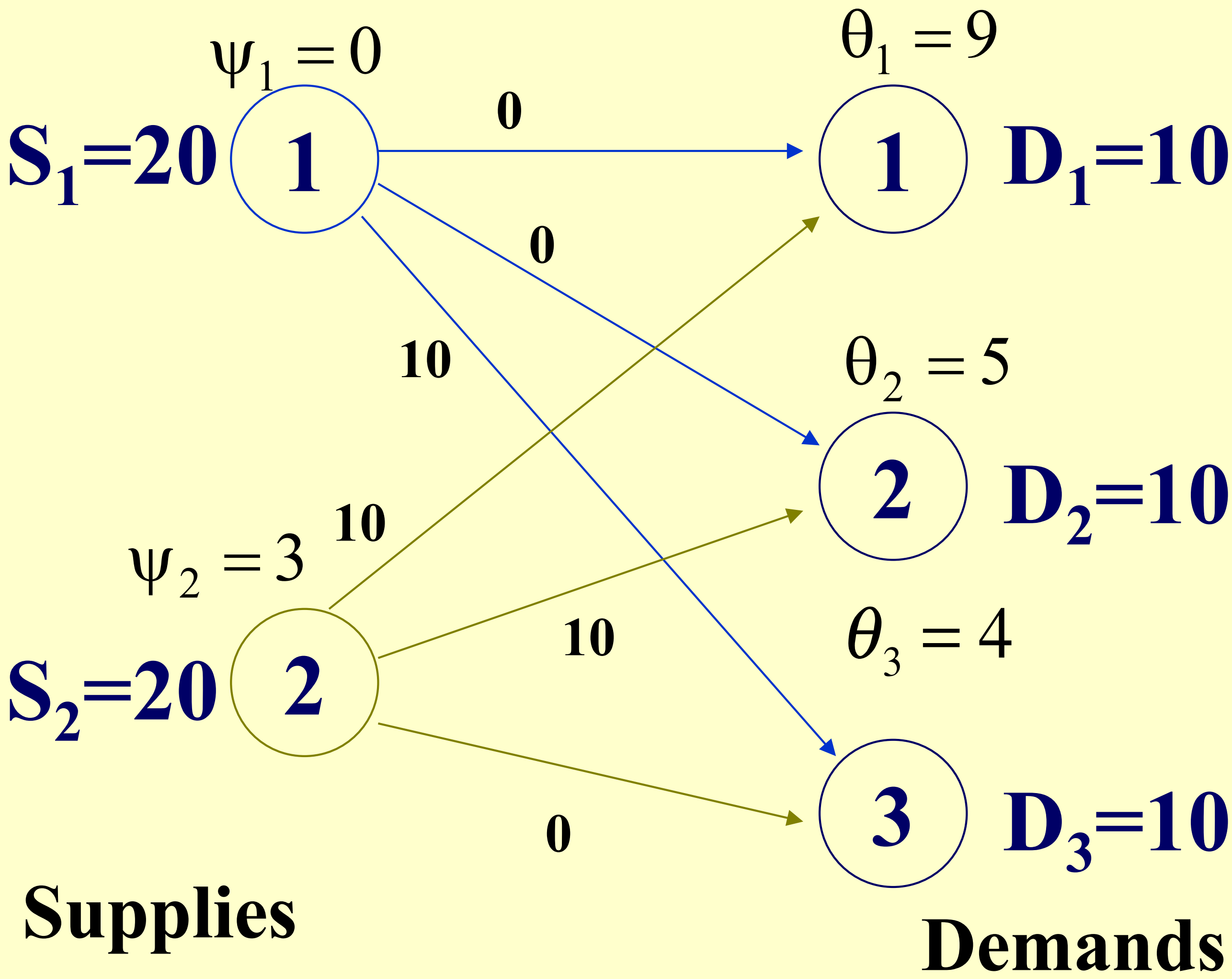
SPE as a Variation on a Transportation Problem (Harker)





- **Want to ship to meet demand at minimal cost**

Spatial Price Equilibrium



Solution:

- **flow on arcs**
- **dual prices at nodes**

Spatial Price Equilibrium

Optimality conditions include conditions of the form

$$\mathrm{c}_{\mathrm{ij}} + \psi_i \geq \theta_j, i = 1, 2, j = 1, 2, 3$$

$$x_{ij} > 0 \Rightarrow \mathrm{c}_{\mathrm{ij}} + \psi_i = \theta_j$$

economic interpretation?

Spatial Price Equilibrium

Remarks:

1. The supply and demand quantities were given as constants, this is less realistic than allowing them to vary as a function of the appropriate prices ($\psi_i, i = 1,2$ for supply, $\theta_j, j = 1,2,3$ for demand) why?

2. Can generalize the optimality conditions stated before using price-dependent supply and demand

Spatial Price Equilibrium

Assume the following (inverse) supply and demand functions:

Supply

$\psi_1(S_1) = S_1 - 20$

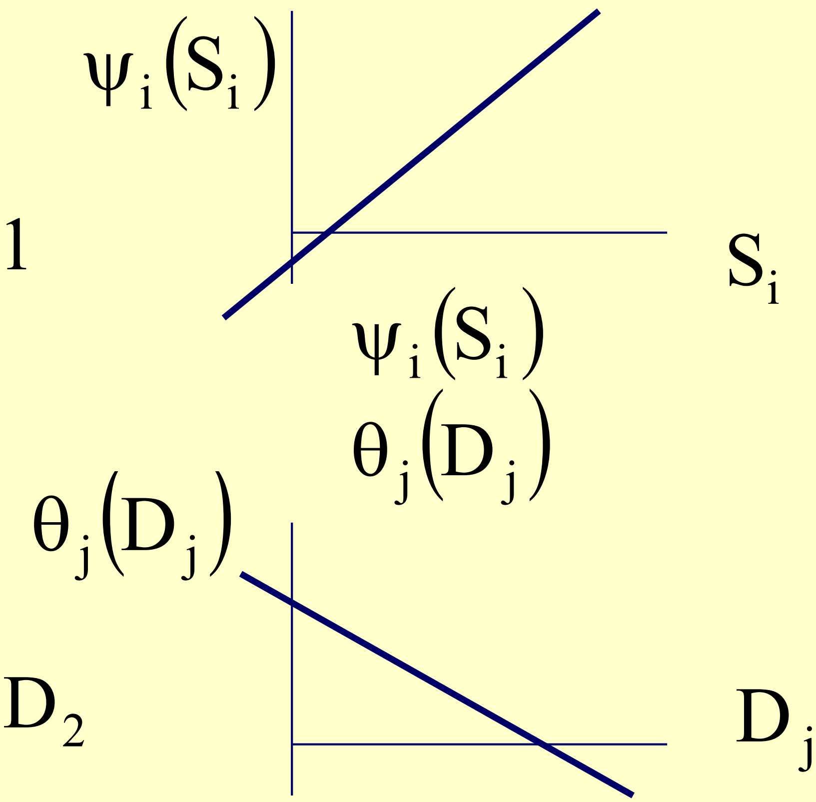
$\psi_2(S_2) = 0.2S_2 - 1$

Demand

$\theta_1(D_1) = 19 - D_1$

$\theta_2(D_2) = 10 - 0.5D_2$

$\theta_3(D_3) = 14 - D_3$



Spatial Price Equilibrium

Complete Optimality Conditions

$$c_{ij} + \psi_i\left(S_i\right) \geq \theta_j\left(D_j\right), x_{ij} \geq 0, i = 1,2, j = 1,2,3$$

$$x_{ij} > 0 \Rightarrow c_{ij} + \psi_i\left(S_i\right) = \theta_j\left(D_j\right)$$

$$\text{with } S_i \equiv \sum_{j=1}^{3} x_{ij}, i = 1,2, D_j \equiv \sum_{i=1}^{2} x_{ij}, j = 1,2,3$$

Why the above generalized slightly may not be solvable by a suitable optimization problem (Principle of Symmetry).

- **Claim: This is an instance of a mixed NCP, why?**

Spatial Price Equilibrium

Spatial Price Equilibrium is an example of a mixed NCP

$$\mathrm{c}_{\mathrm{ij}} + \psi_i\left(S_i\right) \geq \theta_j\left(D_j\right), x_{ij} \geq 0, i = 1,2, j = 1,2,3$$

$$x_{ij} > 0 \Rightarrow \mathrm{c}_{\mathrm{ij}} + \psi_i\left(S_i\right) = \theta_j\left(D_j\right)$$

$$S_i = \sum_{j=1}^{3} x_{ij}, i = 1,2, D_j = \sum_{i=1}^{2} x_{ij}, j = 1,2,3$$

with the following function F

$$F(x_{ij}, i = 1,2, j = 1,2,3)$$

$$= \left(\mathrm{c}_{\mathrm{ij}} + \psi_i\left(\sum_{j=1}^{3} x_{ij}\right) - \theta_j\left(\sum_{i=1}^{2} x_{ij}\right), i = 1,2, j = 1,2,3 \right)$$

Spatial Price Equilibrium with Equity-Enforcing

In this third example, the data from Example #3b are used but an additional constraint of the if-then type is used to demonstrate the flexibility of the proposed DC-MCP approach. Since the solution to Example #3b shows that the energy supply node 4 has no flow from it, a supply planner trying to better balance the supply-demand network could add constraints on top of the equilibrium conditions for better equity between the supply nodes. Consider the following logic that such an energy planner might use to enforce some kind of equity in the network:

$$\text{if } \sum_j x_{ij} < \delta_i \text{ then } \sum_j x_{ij} \geq 0.25 \sum_i \sum_j x_{ij}, \ \forall i$$

where δ_i is some minimum contractual threshold for supply guranteed to supply node i. This if-then condition says that if the SPE flow is less than the contractual minimum, then the ith energy supply node gets at least $\frac{1}{4} = 25\%$ of the total flows. Such conditions are implemented by adding the following constraints where the M_i are positive constants to be chosen ($M_i = 1000$ was

Spatial Price Equilibrium with Equity-Enforcing Constraints

$$\delta_i - \sum_j x_{ij} \leq \widehat{b}_i M_i, i = 1, 2, 3, 4$$

$$-\sum_j x_{ij} + 0.25 \sum_i \sum_j x_{ij} \leq M_i \left(1 - \widehat{b}_i\right), i = 1, 2, 3, 4$$

$$\widehat{b}_i \in \{0, 1\}, i = 1, 2, 3, 4$$

Spatial Price Equilibrium with Equity-Enforcing Constraints

Using $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 3$, the following is the DC-LCP solution reported by GAMS.

$$x_{ij} = \begin{bmatrix} \mathbf{i/j} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \\ \mathbf{1} & 0 & 12 \text{ (was 15)} & 0 & 0 & 20 \\ \mathbf{2} & 20 & 10 & 0 & 0 & 0 \\ \mathbf{3} & 0 & 0 & 10 & 10 & 15 \\ \mathbf{4} & 0 & 3 \text{ (was 0)} & 0 & 0 & 0 \end{bmatrix}$$

This output shows that only two flows were minimally affected: x_{12} and x_{42} to enforce these equity constraints while at the same time minimizng the deviation from complementarity and preserving integer flows.

Extra Slides

Spatial Price Equilibrium with Equity-Enforcing Constraints

Thus, (12) with integer restrictions on a subset of the flows x_{ij} is an instance of a DC-MCP. The following sample SPE with $i = 1, \ldots, 4$ supply nodes and $j = 1, \ldots, 5$ demand nodes is taken from Chapter 4 of [15]. As given in [15], the (rounded) reported solution is

$$x_{ij} = \begin{bmatrix} \mathbf{i/j} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \\ \mathbf{1} & 0 & 15 & 0 & 0 & 5 \\ \mathbf{2} & 20 & 10 & 0 & 0 & 0 \\ \mathbf{3} & 0 & 0 & 10 & 10 & 15 \\ \mathbf{4} & 0 & 0 & 0 & 0 & 15 \end{bmatrix}$$

These values when non-rounded are actually slightly different and are the following with an associated complementarity sum of $-7.72501E-6$:

Main MINLP to solve DC-MCP Using L1 Norm

Theorem 2 *For each* $i \in I_x$, *assume that* $l_i < u_i$. *Consider any optimal solution* $\left(x^*, y^*, z^{+^*}, z^{-^*}, w^{+^*}, w^{-^*}, b^*, \widetilde{b^*}\right)$ *to (7). Then at most one of* $\left(z_i^{+*}, z_i^{-*}\right)$ *is nonzero and at most one of* $\left(w_i^+, w_i^-\right)$ *is nonzero.*

Main MINLP to solve DC-MCP Using L1 Norm

Theorem 3 *Consider any optimal solution* $\left(x^*, y^*, z^{+^*}, z^{-^*}, w^{+^*}, w^{-^*}, b^*, \widetilde{b^*}\right)$ *to (7) with corresponding optimal objective function value* f^*. *Then,*

$f^* = 0 \Leftrightarrow (x^*, y^*)$ *solve the DC-MCP (1), (5).*

Nonlinear Programs Expressed as Mixed Nonlinear Complementarity Problems

Consider a generic nonlinear program and its resulting KKT conditions

$$\min\ f(x)$$
$$s.t.\ g_i(x) \le 0, i = 1,\ldots,m \quad (u_i)$$
$$h_j(x) = 0,\ j = 1,\ldots,p \quad (v_j)$$

KKT conditions, find $\overline{x} \in R^n, \overline{u} \in R^m, \overline{v} \in R^p\, s.t.$

$$\left\{\begin{array}{l} (i)\nabla f(\overline{x}) + \sum_{i=1}^{m} \overline{u}_i \nabla g_i(\overline{x}) + \sum_{j=1}^{p} \overline{v}_i \nabla h_j(\overline{x}) = 0 \\ (ii) g_i(\overline{x}) \le 0, \overline{u}_i \ge 0, g_i(\overline{x})\overline{u}_i = 0, \text{ for all } i = 1,\ldots,m \\ (iii) h_j(\overline{x}) = 0, \overline{v}_j \text{ free, for all } j = 1,\ldots,p \end{array}\right\}$$

Nonlinear Programs Expressed as Mixed Nonlinear Complementarity Problems

Thus, we get a mixed NCP as follows:

$$F\begin{pmatrix} x \\ u \\ v \end{pmatrix} = \begin{pmatrix} \nabla f(x) + \sum_{i=1}^{m} u_i \nabla g_i(x) + \sum_{j=1}^{p} v_j \nabla h_j(x) \\ -g_i(x), i = 1,\ldots,m \\ h_j(x), j = 1,\ldots,p \\ \end{pmatrix}$$

$$\nabla f(x) + \sum_{i=1}^{m} u_i \nabla g_i(x) + \sum_{j=1}^{p} v_j \nabla h_j(x) = 0 \qquad x \text{ free}$$

$$-g_i(x) \geq 0, i = 1,\ldots,m \qquad u_i \geq 0, \left(-g_i(x)\right) * u_i = 0$$

$$h_j(x) = 0, j = 1,\ldots,p \qquad v_j \text{ free}$$

-Many other examples in energy, see for example Gabriel et al. (2013)

Re-expressing the bounded MCP as the zero of a particular median-related function H Traditional Case

The traditional case for $l_i = 0, u_i = +\infty, i \in I_x, l_j = -\infty, u_j = +\infty, j \in I_y$

When we specify $l_i = 0,\ u_i = +\infty\ \forall i \in I_x$ corresponding to the traditional MCP and make a boundedness assumption, the resulting formulation is more efficient (less binary variables) as discussed next. First, we want to exclude cases 4 and 5:

$$\text{case 4: } l_i \le u_i < x_i - F_i(x,y) \Rightarrow z_i = H_i(x,y) = x_i - u_i$$

$$\text{case 5: } l_i \le u_i = x_i - F_i(x,y) \Rightarrow z_i = H_i(x,y) = x_i - u_i = F_i(x,y)$$

There are several ways to exclude cases 4 and 5. For example, suppose that the following assumption is in force.

Re-expressing the bounded MCP as the zero of a particular median-related function H Traditional Case

Assumption 1 There exists a finite $u_i^{\max} \in R_+$ such that $x_i - F_i(x,y) \leq u_i^{\max}$ for all $x \in R_+^{n_x}$, $y \in R^{n_y}$.

Then, if u_i is selected greater than $u_i^{\max}$, we have

$$x_i - F_i(x,y) \leq u_i^{\max} < u_i$$

so that cases 4 and 5 are not possible. Assumption 1 is mild but rules out functions like $F_i(x,y) = -\frac{1}{x_i}$ where $x_i - F_i(x,y) \to +\infty$ as $x_i \to 0$, which for any finite choice of u_i would not necessarily rule out cases 4 and 5. Another way to exclude cases 4 and 5 is to set $u_i = +\infty$ so that cases 4 and 5's conditions combined

$$l_i \leq u_i = +\infty \leq x_i - F_i(x,y)$$

are never true for finite x, y.[3] Thus, for specificity but without loss of generality, from here on we take $l_x = 0, u_x = +\infty$ so that the resulting three cases are:

Case 1: $x_i - F_i(x,y) < 0 = l_i \leq u_i = \infty \Rightarrow z_i = H_i(x,y) = x_i$
Case 2: $0 = l_i < x_i - F_i(x,y) \leq u_i = \infty \Rightarrow z_i = H_i(x,y) = F_i(x,y)$
Case 3: $0 = l_i = x_i - F_i(x,y) \leq u_i = \infty \Rightarrow z_i = H_i(x,y) = F_i(x,y) = x_i$

Spatial Price Equilibrium with Equity-Enforcing Constraints

The spatial price equilibrium problem (SPE) is a generalization of the classical linear programming transportation problem [23], [16], [15]. In the SPE, given a bipartite network of spatially dispersed supply nodes $i \in I$ and demand nodes $j \in J$ and set of connecting arcs $a \in A = \{(i,j) : i \in I, j \in J\}$ for the resulting complete network, the objective is to determine the vector of nonnegative flows $x = \{x_{ij} : i \in I, j \in J\}$ such that

$$0 \leq \Psi_i \left(\sum_j x_{ij} \right) + c_{ij}\left(x_{ij}\right) - \theta_j \left(\sum_i x_{ij} \right) \perp x_{ij} \geq 0$$